## Chap 4 - Utility

→ We'll use the term "utility" to refer to ~~as~~ a consumer's well-being

→ To put this in the context of consumer preferences from Chap 3, we'll say that bundle $(x_1, x_2)$ gives the consumer more utility than $(y_1, y_2)$ if

$$(x_1, x_2) \succ (y_1, y_2)$$

→ a utility function is a function that assigns a value to every possible consumption bundle.

### Types of Utility functions

→ We can split utility functions into 2 types: ordinal and cardinal

→ Ordinal utility functions assign values to consumption bundles in a way that preserves the consumers preference ordering, but does not give information about the "magnitude" of those preferences.

e.g. If Alex ~~prefer~~ has the following preferences: apples (A) $\succ$ bananas (B) $\succ$ cherries (C), we could have the following ordinal utility function:

| Bundle | Utility$_1$ |
|---|---|
| A | 5 |
| B | 4 |
| C | 3 |

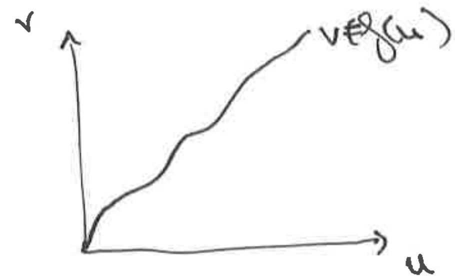
→ ~~but it~~ this preserves the order

→ it doesn't make sense to say that bundle A gives 25% more utility than B b/c we could write down another utility function that also ~~expresses~~ is consistent w/ Alex's prefs

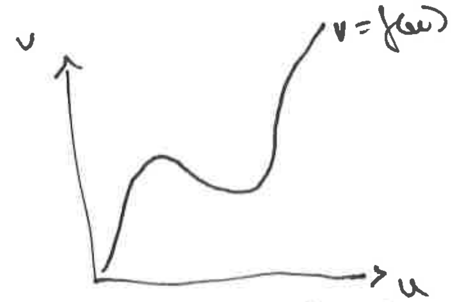
| Bundle | Utility 2 |
|---|---|
| A | 25 |
| B | 16 |
| C | 9 |

→ this utility function also represents
A > B > C

more generally,
→ ordinal utility functions have the property that they ~~are preserve~~ represent the same preference under any monotonic transformation




monotonic ~~trans~~ function $f(u)$




non-monotonic function

→ a monotonic transformation of u is given by applying the monotonic function (i.e. always increasing function to $f(u)$ to u.

e.g. let $f(u) = u \times 2$

| Bundle | u | f(u) |
|---|---|---|
| A | 5 | 10 |
| B | 4 | 8 |
| C | 3 | 6 |

→ same pref ordering = represent same ordinal utility function

→ cardinal utility functions are those that don't just rank preferences but assign specific values to those preferences

→ economists typically don't use cardinal utility functions

→ what does it mean to say you like some bundle twice as much as another?

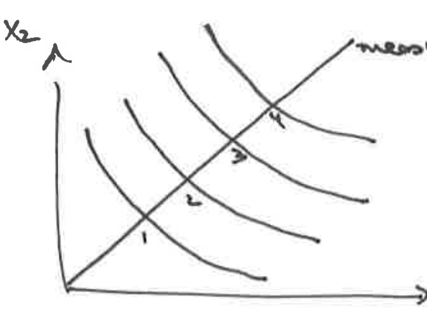
→ you might say you'd give up $1/2$ ~~a little~~ of your entree for 2 times more dessert, but what does it mean to say you like the meal w/ the larger dessert twice as much as the other meal?

Constructing a utility function

→ We ~~should~~ will always be able to define a utility function that represents preferences if our 3 ~~axioms~~ assumption of consumer theory hold. That is, if preferences are:

1) ~~[illegible]~~ Complete
2) Reflexive
3) Transitive

→ The utility function can then be any function that assigns a higher number to a higher indifference curve e.g.




measures distance to the origin

Examples of utility functions

→ ~~these~~ first, we'll start w/ a utility function and show the indiff curves it yields

→ next, we look at indiff curves and find the utility function that fits them

→ recall that an indifference curve is a set of consumption bundles the consumer is indifferent between

→ thus, the utility from any consumption bundle on the indiff curve will be the same

i.e. the set of $(x_1, x_2)$ such that $u(x_1, x_2)$ equals a constant is an indifference curve

→ an so the set of $(x_1, x_2)$ that result in $u(x_1, x_2)$ equal to another constant ~~represent~~ another indifference curve

→ as an example, consider $u(x_1, x_2) = x_1 x_2$

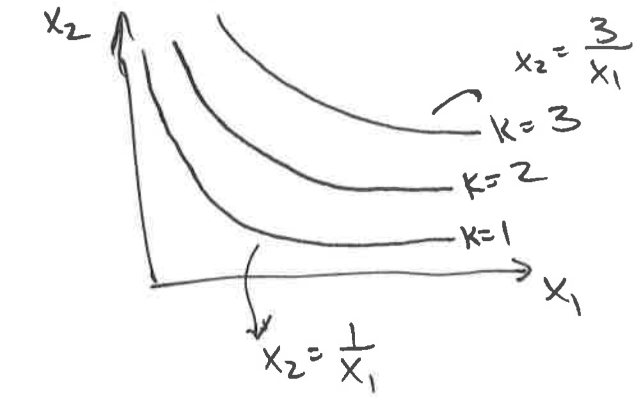
→ to solve for ~~the~~ an indifference curve, set the utility from $(x_1, x_2)$ equal to some constant, $k$:

$$u(x_1, x_2) = x_1 x_2 = k$$

$$\Rightarrow \underline{x_2 = \frac{k}{x_1}}$$

this describes the indifference curve

Graphically




→ now consider ~~$u$~~$(x_1, x_2) = x_1^2 x_2^2$

$$v(x_1, x_2) = x_1^2 x_2^2 = (x_1 x_2)^2 = u(x_1, x_2)^2$$

⇒ $v(x_1, x_2)$ a monotonic transformation of $u(x_1, x_2)$

→ thus will represent the same prefs and have the same indiff curves

To see:

indiff curve given by $v(x_1, x_2) = c$, where $c$ is some constant

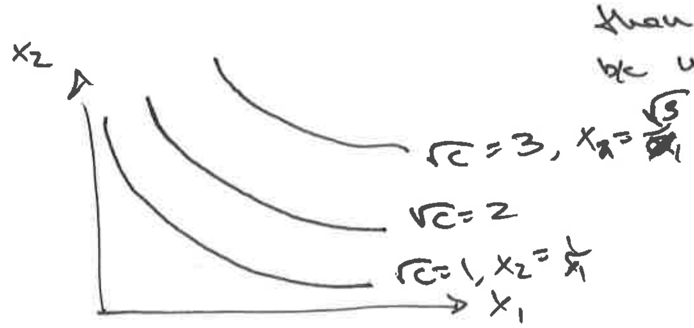
$$\Rightarrow v(x_1, x_2) = x_1^2 x_2^2 = c$$

$$\Rightarrow x_2^2 = \frac{c}{x_1^2}$$

$$\Rightarrow x_2 = \frac{\sqrt{c}}{x_1}$$

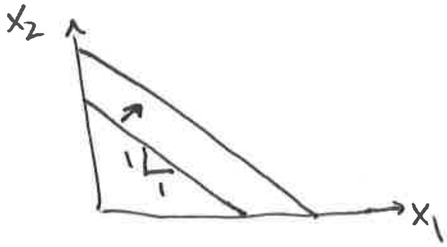
so if $\sqrt{c} = k$ (or $c = k^2$)

then same – and ok to do this b/c utility only ordinal – value



→ Now lets consider some preferences whose indiff curves we know and see if we can recover the utility function that yields them:

→ Perfect Substitutes:




does $u(x_1, x_2) = x_1 + x_2$ represent this?

→ indiff curve given by $x_1 + x_2 = k$

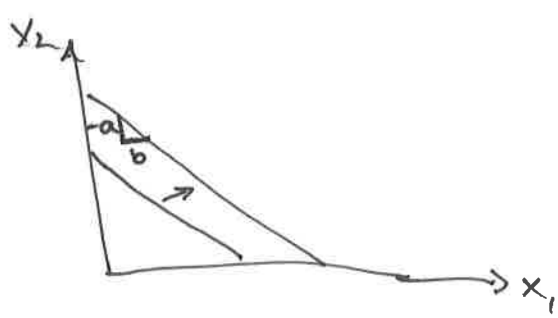
$\Rightarrow x_2 = \underbrace{k - x_1}$

intercept of $k$,
slope $= -1$ ✓

→ also, as $x_1$ or $x_2 \uparrow$, move to higher indiff curve, which are more preferred bundles

→ so $u(x_1, x_2) = x_1 + x_2$ represents these preferences

→ In general, perfect subs are represented by: $u(x_1, x_2) = ax_1 + bx_2$, where $a$ and $b$ measure the "value" of goods 1 and 2 to the consumer:




Indiff curves given by $ax_1 + bx_2 = k$

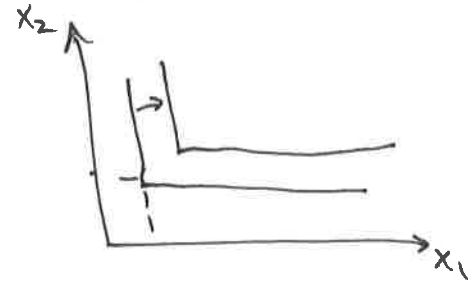
$\Rightarrow x_2 = \underbrace{\frac{k}{b} - \frac{a x_1}{b}}$

intercept $= \frac{k}{b}$
slope $= \frac{-a}{b}$

→ of course any monotonic transform of this would work

→ Perfect complements

→ recall indiff curves = "L-shaped":



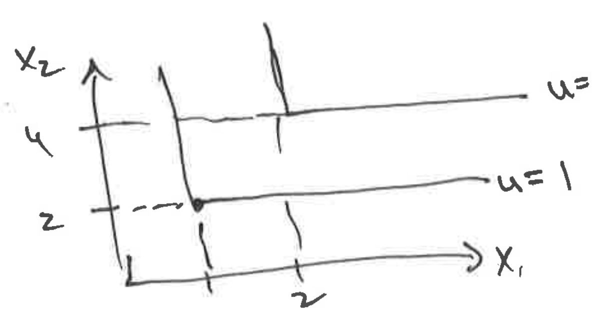

→ consider case of shoes: what matters for utility is the number of complete pairs

→ this is given by the minimum number of right or left shoes

→ here $u(x_1, x_2) = \min\{x_1, x_2\}$ would work (or any monotonic transform)

→ in general, for perfect complements, $u(x_1, x_2) = \min\{a x_1, b x_2\}$ describes these preferences, where a and b indicate the proportions in which the goods are consumed

e.g. $a = 1$ tbs, $x_1$ = jelly
$b = 2$ tbs, $x_2$ = peanut butter

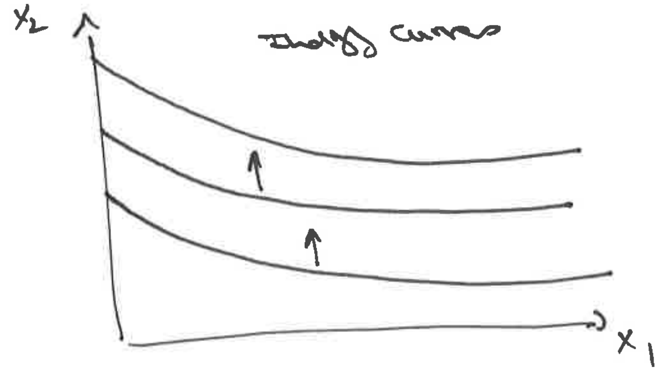



→ careful w/ a and b
→ 2 tbs PB to 1 tbs jelly means $b = \frac{1}{2} = \frac{\text{tbs jelly}}{\text{tbs PB}}$

## Quasilinear Preferences



→ these prefs are represented by indiff curves that are shifted vertically:




→ so eq'n for IC is: $x_2 = k - v(x_1)$ where $k$ is a constant

→ solving for $u(x_1, x_2) = k$ we find

$$u(x_1, x_2) = v(x_1) + \underbrace{x_2}_{\text{linear in } x_2}$$

→ hence name "quasi-linear" (i.e. partly linear)

→ used often b/c easy to work with

## Cobb-Douglas Preferences

→ one very common utility function is the Cobb-Douglas utility function

→ This function is of the form:

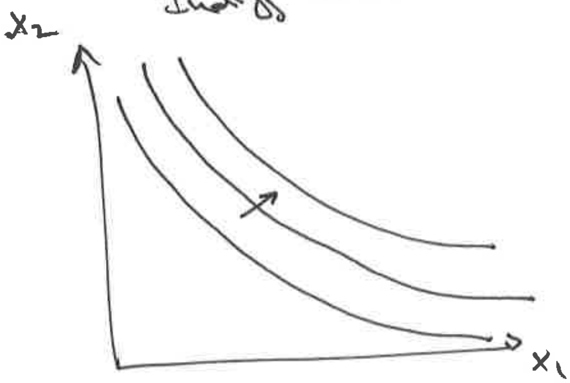
$$u(x_1, x_2) = x_1^c x_2^d$$

where $c, d > 0$

→ $c, d$ describe curvature of indiff curves

→ a nice property of this utility function is that it will ensure at least some of each good is consumed.

Indiff curves




→ just like w/ any other utility function, a monotonic transformation of the Cobb-Douglas function will represent the same preferences.

→ e.g could do:

$$v(x_1, x_2) = u(x_1, x_2)^{\frac{1}{c+d}} = \left(x_1^c x_2^d\right)^{\frac{1}{c+d}}$$

$$= x_1^{\frac{c}{c+d}} x^{\frac{d}{c+d}}$$

now define $a = \frac{c}{c+d}$

We thus have:

$$v(x_1, x_2) = x_1^{a} x_2^{1-a}$$

→ this means we can always take a monotonic transformation of the Cobb-Douglas function where the exponents sum to one.

$$a + 1 - a = 1$$

→ we'll see later on that this will have a useful interpretation.

Marginal Utility

→ Marginal utility will define the rate of change in the utility function ~~of~~ w.r.t. a change in the consumption of a good.

→ marginal just means derivative.

→ This will be a key concept.

→ Formally, we define the marginal utility of good 1

as:

$$MU_1 = \lim_{\Delta x \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1} = \frac{\partial u(x_1, x_2)}{\partial x_2}$$

→ partial derivative of $u(x_1, x_2)$ w.r.t. $x_1$

→ analogous for $MU_2$

Example: $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{1}{2} x_1^{1/2 - 1} x_2^{1/2}$$
$$= \tfrac{1}{2} x_1^{-1/2} x_2^{1/2}$$
$$= \frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2}$$

$$MU_2 = \frac{\partial u(x_1, x_2)}{\partial x_2} = \frac{1}{2} x_1^{1/2} x_2^{1/2 - 1}$$
$$= \tfrac{1}{2} x_1^{1/2} x_2^{-1/2}$$
$$= \frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2}$$

## Marginal Utility and the MRS

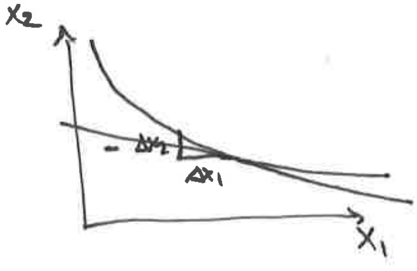
→ Recall our defn of the MRS:

→ MRS = slope of indiff curve at a given point of consumption

→ tells us how much of one good we'd need to get for giving up consumption of another.

→ so the slope of the IC is negative

→ $MRS = -\frac{\Delta x_2}{\Delta x_1}$




→ the fact that the MRS is the slope of the IC tells us that a derivative will be involved.

→ 2 ways to solve for the MRS:

1. Using the total differential of the utility function:

$$\underbrace{du}_{} = 0$$

says that utility doesn't change, which is true along ~~the~~ an indifference curve.

$$du = \underbrace{\frac{\partial u(x_1, x_2)}{\partial x_1} dx_1}_{\underbrace{MU_1 dx_1}_{\substack{=\text{ change in utility} \\ \text{for a change} \\ \text{in } x_1}}} + \underbrace{\frac{\partial u(x_2, x_2)}{\partial x_2} dx_2}_{MU_2 dx_2} = 0$$

→ use this to solve for $MRS = \frac{\Delta x_2}{\Delta x_1} = \frac{dx_2}{dx_1}$ :

$$\frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_2, x_2)}{\partial x_2} dx_2 = 0$$

$$\Rightarrow \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 = -\frac{\partial u(x_1, x_2)}{\partial x_2} dx_2$$

$$\Rightarrow \frac{\partial u(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial u(x_1, x_2)}{\partial x_2} = \underbrace{\frac{dx_2}{dx_1}}_{= MRS}$$

$\Rightarrow$ MRS = the negative of the ratio of marginal utilities

2 Using implicit functions.

an indiff curve is defined by:

~~$u(x_1, x_2) = k$~~

$\Rightarrow$ $u(x_1, x_2(x_1)) = k$

$\Rightarrow$ $x_2(x_1)$ gives the value of $x_2$ that ensures the utility is equal to $k$ given $x_1$

$\rightarrow$ now take the derivative of both sides w.r.t. $x_1$:

$$\underbrace{\frac{\partial u(x_1, x_2(x_1))}{\partial x_1} + \frac{\partial u(x_1, x_2(x_1))}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1}}_{\text{by chain rule}} = \underbrace{\frac{\partial k}{\partial x_1}}_{=0}$$

$$\Rightarrow \frac{\partial u(x_1, x_2(x_1))}{\partial x_1} + \frac{\partial u(x_1, x_2(x_1))}{\partial x_2}\underbrace{\frac{\partial x_2(x_1)}{\partial x_1}}_{\text{solve for this, it is the MRS}} = 0$$

$$\Rightarrow \frac{\partial u(x_1, x_2(x_1))}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial u(x_1, x_2(x_1))}{\partial x_1}$$

$$\Rightarrow \frac{\partial x_2(x_1)}{\partial x_1} = \underbrace{-\frac{\partial u(x_1, x_2(x_1))}{\partial x_1} \Big/ \frac{\partial u(x_1, x_2(x_1))}{\partial x_2}}_{\text{ratio of MU}}$$

→ So the ~~ratio~~ MRS = the negative of the ratio of marginal utilities

→ i.e. $MRS = \frac{MU_1}{MU_2}$

Example: Cobb-Douglas utility

$$u(x_1, x_2) = x_1^c x_2^d$$

$$MRS = -\frac{MU_1}{MU_2}$$

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} = c x_1^{c-1} x_2^d$$

$$MU_2 = \frac{\partial u(x_1, x_2)}{\partial x_2} = d x_1^c x_2^{d-1}$$

$$\Rightarrow MRS = -\left( \frac{c x_1^{\cancel{c}-1} \cancel{x_2}^{\cancel{d}}}{d \cancel{x_1}^{\cancel{c}} x_2^{\cancel{d}-1}} \right)$$

$$= -\left( \frac{c x_1^{-1}}{d x_2^{-1}} \right) = -\left( \frac{c x_2}{d x_1} \right)$$

→ Now that we know the mathematical relationships we can easily show that preferences, ~~define~~ which are defined by IC, ~~don't~~ remain unchanged under monotonic transformations of the utility function:

$$\text{e.g. define } v(x_1, x_2) = u(x_1,x_2)^2 = \left(x_1^c x_2^d\right)^2 = x_1^{2c} x_2^{2d}$$

$$MRS = -\frac{\partial v(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial v(x_1, x_2)}{\partial x_2}$$

$$= -\left(\frac{2c\, x_1^{2c-1}\, x_2^{2d}}{2d\, x_1^{2c}\, x_2^{2d-1}}\right) = -\left(\frac{\cancel{2}c\, x_1^{-1}}{\cancel{2}d\, x_2^{-1}}\right)$$

$$= -\frac{c}{d}\frac{x_2}{x_1}$$

same as MRS for $u(x_1, x_2)$

Applying this

→ Varian gives a nice example of how economists have applied utility functions

→ e.g. assume utility from transportation given as

$$u(TW, TT, C) = \beta_1 TW + \beta_2 TT + \beta_3 C$$

where TW = total walking time (minutes)
TT = total travel time (minutes)
C = trip cost ($'s)

→ w/ econometric techniques, one can estimate this function w/ obs. of peoples choices of transportation method

→ the value of $U(TW, TT, C)$ is not important, but the MRS is important and informative

$$MRS_{\cancel{TW}TT,C} = -\frac{MU_{TT}}{MU_C} = -\frac{\beta_2}{\beta_3}$$

Domenich & McFadden finds
$$\beta_1 = -0.147$$
$$\beta_2 = -0.0411$$
$$\beta_3 = -2.24$$

$$\Rightarrow MRS_{TT,C} = \frac{-0.0411}{-2.24} = 0.0183$$

→ willing to pay 0.0183 dollars per minute travel time reduced

→ this is informative for transport policy (among other things)